Fundamentals of Finite Elements Method

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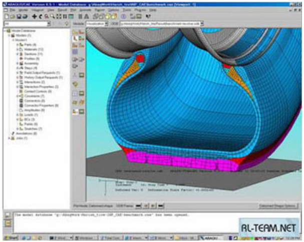
Computer simulation

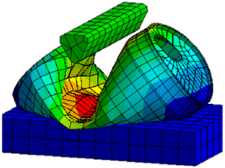
Computer simulation is the reproduction of the behaviour of a system using a computer to simulate the outcomes of a mathematical model associated with said system. Computer simulations have become a useful tool for the mathematical modelling of many natural systems in physics, astrophysics, climatology, chemistry, biology, engineering, etc.

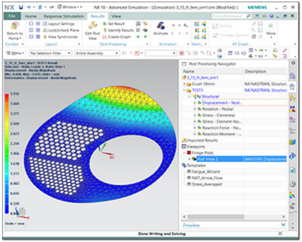
Computer simulations are realized by running computer programs implemented **computational methods**. Programs can be either small, running on small devices, or large-scale programs that run for hours or days on network-based groups of computers (cluster computers).

The scale of events being simulated by computer simulations has far exceeded anything possible using traditional paper-and-pencil mathematical modelling. Simulation can be used to explore and to estimate the performance of systems too complex for analytical solutions.

Computer programs enable simulation of kinematics or dynamics of the system, analysis of heat flow and mass flow, stresses and other characteristics of the designed product. This allows for a significant acceleration of the design process and - above all - for reduction of design costs.

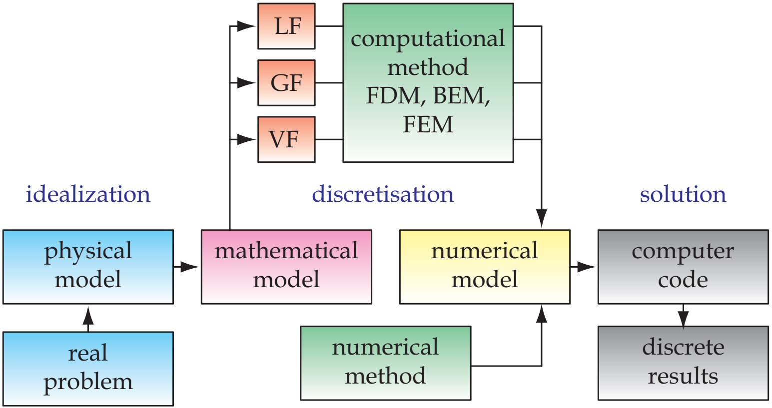






Examples of software packages that implement the finite element method: **Abaqus**, **ADINA**, **NX**, **FEMAP**, **CalculiX**, **LS-DYNA**.

Analysis process



In most cases, the **mathematical model** of a **real problem** is derived from the **physical phenomenon**. The physical model is the set of assumptions with some simplifications made to the real problem. The simplifications on the physical assumptions lead to the less appropriate mathematical model, but at the same time the one that is simpler to solve exactly.

Mostly, a mathematical modeln is expressed by the **algebraic equation** or **differential equation** (set of equations) completed with boundary conditions that need to be solved.

The subject of computer simulation is concerned with devising **computational methods** for **solution** or mostly **approximating the solution** to mathematically expressed problem.

Computational method consists of a sequence of algebraic and logical operation that produces the approximation to the mathematical problem thus the method can be employed on digital computers.

Solutions of a mathematical model equations are functions or values of these functions. In case of computational methods values of functions are computed for **fixed values of parameters** and **fixed values of independent variables**.

Mathematical model formulated as set of differential equations correspond to the so-called **local formulation**.

Mathematical model formulated as functionals associated with the whole structure are used in the **global formulation**. In such a case function of kinetic energy and **potential energy** are often used as well as their combination.

The computational methods for solution of systems most often deal with initial or boundary value problems.

- Methods for solving **initial value** problems (IVP):
  - Euler methods.
  - Runge-Kutta type of methods.
- Methods for solving **boundary value** problems (BVP):
  - Finite difference method.
  - Boundary element method.
  - Finite element method.

The finite element method (FEM), is a method for solving problems of engineering and mathematical physics, formulated as differential equation or functional.

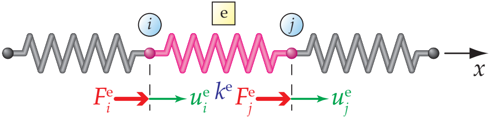
The method approximates the unknown function over the domain. To solve the problem, it subdivides a large system into smaller, simpler parts that are called **finite elements**. The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem. FEM uses variational methods from the calculus of variations to approximate a solution by minimizing an associated error function.

What is FEM for an engineer?

FEM is a numerical method that allows finding approximate and discrete functions to solve the boundary problem. In the case of the linear theory of elasticity (solid state mechanics), displacements are the solution of the problem, so we can also determine strains and stresses. The main advantage of the method is the ability to obtain results for complex structures, which it is not possible to perform analytical calculations.

Spring

The model of simplest finite element is spring with stiffness k.



Basic finite element properties:

- **nodes**: i, j (**local**, related with finite element),
- **stiffness**: k^e,
- nodal **displacements**: u_i^e, u_j^e (**local**),
- nodal **forces**: F_i^e, F_j^e (**local**).

The relation of force and displacement has the form

$$F^{e} = k^{e} \cdot \Delta u^{e},$$
$$\Delta u^{e} = u_j^{e} - u_i^{e},$$

using the equilibrium condition, we can write

$$F_i^{e} + F_j^{e} = 0 \Rightarrow F_j^{e} = -F_i^{e} = F^{e}.$$

Consider the equilibrium of forces for the spring. At nodes we have

$$F_i^{e} = -F^{e} = -k^{e}\left(u_j^{e} - u_i^{e}\right) = k^{e} \cdot u_i^{e} - k^{e} \cdot u_j^{e},$$
$$F_j^{e} = F^{e} = k^{e}\left(u_j^{e} - u_i^{e}\right) = -k^{e} \cdot u_i^{e} + k^{e} \cdot u_j^{e}.$$

In the matrix form

$$\begin{bmatrix} k^{e} & -k^{e} \\ -k^{e} & k^{e} \end{bmatrix} \begin{bmatrix} u_i^{e} \\ u_j^{e} \end{bmatrix} = \begin{bmatrix} F_i^{e} \\ F_j^{e} \end{bmatrix},$$

or

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{F}^e,$$

where:

- $\mathbf{K}^e$ – finite element stiffness matrix,
- $\mathbf{u}^e$ – element nodal displacement vector,
- $\mathbf{F}^e$ – element nodal force vector,

$$\mathbf{K}^e \equiv \begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix},$$

$$\mathbf{u}^e \equiv \begin{bmatrix} u_i^e \\ u_j^e \end{bmatrix},$$

$$\mathbf{F}^e \equiv \begin{bmatrix} F_i^e \\ F_j^e \end{bmatrix},$$

$$\mathbf{K}^{\mathrm{e}} = \begin{bmatrix} k^{\mathrm{e}} & -k^{\mathrm{e}} \\ -k^{\mathrm{e}} & k^{\mathrm{e}} \end{bmatrix}.$$

Notes:

- stiffness matrix $\mathbf{K}^{\mathrm{e}}$ is symmetric $\mathbf{K}^{\mathrm{e}} = (\mathbf{K}^{\mathrm{e}})^{\mathrm{T}}$,
- stiffness matrix is singular, determinant is equal to zero

$$|\mathbf{K}^{\mathrm{e}}| = k^{\mathrm{e}} \cdot k^{\mathrm{e}} - (-k^{\mathrm{e}})(-k^{\mathrm{e}}) = (k^{\mathrm{e}})^2 - (k^{\mathrm{e}})^2 = 0.$$

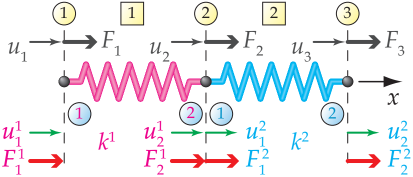
Can we solve that equations?

All you have to do is provide only one boundary condition for the displacement u^{e}, to solve the equations.

For $u_i^{\mathrm{e}} = 0$ (restrained end of the beam), to $u_j^{\mathrm{e}} = \dfrac{F^{\mathrm{e}}}{k^{\mathrm{e}}}$.

Spring System

Derivation of equilibrium equations for the spring system.



Equilibrium equations for each elements (**local**)

$$\begin{bmatrix} k^1 & -k^1 \\ -k^1 & k^1 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1 \end{bmatrix},$$

$$\begin{bmatrix} k^2 & -k^2 \\ -k^2 & k^2 \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix} = \begin{bmatrix} F_1^2 \\ F_2^2 \end{bmatrix}.$$

Global displacements are

$$u_1 = u_1^1,$$
$$u_2 = u_2^1 = u_1^2,$$
$$u_3 = u_2^2.$$

Consider the equilibrium of forces ant each nodes

$$F_1 = F_1^1,$$
$$F_2 = F_2^1 + F_1^2,$$
$$F_3 = F_2^2,$$

that is

$$F_1 = k^1 u_1 - k^1 u_2,$$
$$F_2 = -k^1 u_1 + k^1 u_2 + k^2 u_2 - k^2 u_3,$$
$$F_3 = -k^2 u_2 + k^2 u_3.$$

After transformation

$$F_1 = k^1 u_1 - k^1 u_2,$$
$$F_2 = -k^1 u_1 + \left(k^1 + k^2\right) u_2 - k^2 u_3,$$
$$F_3 = -k^2 u_2 + k^2 u_3.$$

In matrix form

$$\begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 + k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \begin{bmatrix} u_1 \equiv u_1^1 \\ u_2 \equiv u_2^1 = u_1^2 \\ u_3 \equiv u_2^2 \end{bmatrix} = \begin{bmatrix} F_1 \equiv F_1^1 \\ F_2 \equiv F_2^1 + F_1^2 \\ F_3 \equiv F_2^2 \end{bmatrix},$$

or

$$\mathbf{Ku} = \mathbf{F},$$

where:

- **K** – global stiffness matrix (structure matrix),
- **u** – global displacement vector,
- **F** – global force vector.

An alternative way, „enlarging" the stiffness matrices for element 1

$$\begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1 \\ 0 \end{bmatrix},$$

$$\Updownarrow$$

$$\begin{cases} k^1 u_1 - k^1 u_2 + 0 \cdot u_3 = F_1^1 \\ -k^1 u_1 + k^1 u_2 + 0 \cdot u_3 = F_2^1 \\ 0 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3 = 0 \end{cases},$$

and for element 2

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_1^2 \\ F_2^2 \end{bmatrix}.$$

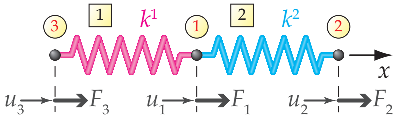
Adding the two matrix equations (*superposition*), we have

$$\left(\begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ F_1^2 \\ F_2^2 \end{bmatrix},$$

this is the same equations we derived by using the force equilibrium concept.

Above operation assembly **local** (for single finite element) equilibrium equations to one **global** (for all finite elements) system of equilibrium equations is called **aggregation**.

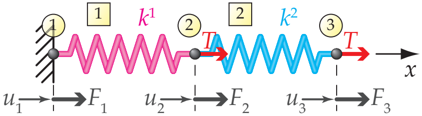
Notice, that the way of **numbering nodes** affects the form of the equilibrium equations.



$$\mathbf{K}^1 = \begin{bmatrix} k^1 & 0 & -k^1 \\ 0 & 0 & 0 \\ -k^1 & 0 & k^1 \end{bmatrix}, \quad \mathbf{K}^2 = \begin{bmatrix} k^2 & -k^2 & 0 \\ -k^2 & k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} k^1 + k^2 & -k^2 & -k^1 \\ -k^2 & k^2 & 0 \\ -k^1 & 0 & k^1 \end{bmatrix} \begin{bmatrix} u_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_3 \\ F_1 \\ F_2 \end{bmatrix}.$$

Spring system with boundary and load condition.



„Enlarge" equilibrium equations in matrix form has form

$$\begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_1^2 \\ F_2^2 \end{bmatrix}.$$

After aggregation, assuming $u_1 = 0$ and $F_1 = F_2 = T$ we have

$$\begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 + k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \begin{bmatrix} u_1 \equiv 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \equiv T \\ F_3 \equiv T \end{bmatrix},$$

$$\Updownarrow$$

$$\begin{cases} k^1 \cdot 0 - k^1 u_2 + 0 \cdot u_3 = F_1 \\ -k^1 \cdot 0 + (k^1 + k^2) u_2 - k^2 u_3 = T \\ 0 \cdot 0 - k^2 u_2 + k^2 u_3 = T \end{cases}.$$

which reduces to (only primary variables)

$$\begin{bmatrix} k^1 + k^2 & -k^2 \\ -k^2 & k^2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T \\ T \end{bmatrix},$$

$$\Updownarrow$$

$$\begin{cases} (k^1 + k^2) u_2 - k^2 u_3 = T \\ -k^2 u_2 + k^2 u_3 = T \end{cases}.$$

After some algebra we have

$$\begin{cases} k^1 u_2 = 2T \\ -k^2 u_2 + k^2 u_3 = T \end{cases},$$

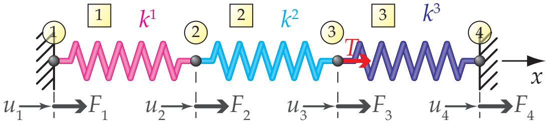
and

$$-k^1 u_2 = F_1.$$

Solving the equations, we obtain the displacement and the reaction force

$$u_2 = \frac{2T}{k^1}, \quad u_3 = \frac{2T}{k^1} + \frac{T}{k^2}, \quad F_1 = -2T.$$

Self study example - system of three springs.



- spring stiffness:
  $k^1 = 100\ ^{\text{N}}/_{\text{mm}}$,
  $k^2 = 200\ ^{\text{N}}/_{\text{mm}}$,
  $k^3 = 100\ ^{\text{N}}/_{\text{mm}}$,
- force $T = 500$ N.

Elements stiffness matrices

$$\mathbf{K}^1 = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}, \quad \mathbf{K}^2 = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}, \quad \mathbf{K}^3 = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix},$$

global stiffness matrix

$$\mathbf{K} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100+200 & -200 & 0 \\ 0 & -200 & 200+100 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix}.$$

Global equilibrium equations

$$\begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} u_1 \equiv 0 \\ u_2 \\ u_3 \\ u_4 \equiv 0 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \equiv 0 \\ F_3 \equiv 500 \\ F_4 \end{bmatrix}.$$

Equations for determine primary variables

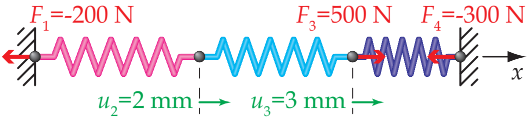
$$\begin{bmatrix} 300 & -200 \\ -200 & 300 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 500 \end{bmatrix},$$

and secondary variables

$$\begin{cases} -100 u_2 = F_1 \\ -100 u_3 = F_4 \end{cases}.$$

Solving the equations, we obtain the displacements and forces

$$u_2 = 2\,\text{mm}, \quad u_3 = 3\,\text{mm}, \quad F_1 = -200\,\text{N} \quad F_4 = -300\,\text{N}.$$



Sum of loads and reactions is equal to zero

$$T + F_1 + F_4 = 0 \Rightarrow 500 - 200 - 300 = 0.$$

The FE equations for the spring 2 is

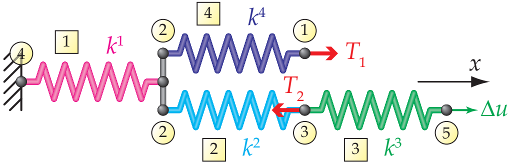
$$\mathbf{K}^2\mathbf{u}^2 = \mathbf{F}^2,$$

$$\begin{bmatrix} k^2 & -k^2 \\ -k^2 & k^2 \end{bmatrix} \begin{bmatrix} u_2 \equiv u_1^2 \\ u_3 \equiv u_2^2 \end{bmatrix} = \begin{bmatrix} F_1^2 \\ F_2^2 \end{bmatrix}.$$

We can calculate the spring forces as

$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} F_1^2 \\ F_2^2 \end{bmatrix} \Rightarrow \begin{bmatrix} F_1^2 \\ F_2^2 \end{bmatrix} = \begin{bmatrix} -200 \\ 200 \end{bmatrix}.$$

Self study example - find the global stiffness matrix for the arbitrarily numbered nodes and elements.



First we construct the **element connectivity table** which specifies the **global** node numbers corresponding to the **local** node numbers for each element.

Element	Node $i = 1$	Node $j = 2$
1	4	2
2	2	3
3	3	5
4	2	1

Element	Node $i = 1$	Node $j = 2$
1	4	2
2	2	3
3	3	5
4	2	1

Then we can write the element stiffness matrices as follows

$$\mathbf{K}^1 = \begin{matrix} & 4 & 2 & \\ 4 & k^1 & -k^1 & 1 \\ 2 & -k^1 & k^1 & 2 \\ & 1 & 2 & \end{matrix}, \qquad \mathbf{K}^2 = \begin{matrix} & 2 & 3 & \\ 2 & k^2 & -k^2 & 1 \\ 3 & -k^2 & k^2 & 2 \\ & 1 & 2 & \end{matrix},$$

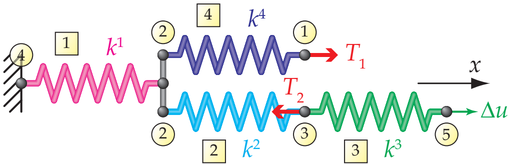
$$\mathbf{K}^3 = \begin{matrix} & 3 & 5 & \\ 3 & k^3 & -k^3 & 1 \\ 5 & -k^3 & k^3 & 2 \\ & 1 & 2 & \end{matrix}, \qquad \mathbf{K}^4 = \begin{matrix} & 2 & 1 & \\ 2 & k^4 & -k^4 & 1 \\ 1 & -k^4 & k^4 & 2 \\ & 1 & 2 & \end{matrix}.$$

$$\mathbf{K}^1 = \begin{array}{c} \\ 4 \\ 2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} 4 & 2 \end{array} \\ \left[\begin{array}{cc} k^1 & -k^1 \\ -k^1 & k^1 \end{array}\right] \\ \begin{array}{cc} 1 & 2 \end{array} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array}, \qquad \mathbf{K}^2 = \begin{array}{c} \\ 2 \\ 3 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \left[\begin{array}{cc} k^2 & -k^2 \\ -k^2 & k^2 \end{array}\right] \\ \begin{array}{cc} 1 & 2 \end{array} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array},$$

$$\mathbf{K}^3 = \begin{array}{c} \\ 3 \\ 5 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} 3 & 5 \end{array} \\ \left[\begin{array}{cc} k^3 & -k^3 \\ -k^3 & k^3 \end{array}\right] \\ \begin{array}{cc} 1 & 2 \end{array} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array}, \qquad \mathbf{K}^4 = \begin{array}{c} \\ 2 \\ 1 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} 2 & 1 \end{array} \\ \left[\begin{array}{cc} k^4 & -k^4 \\ -k^4 & k^4 \end{array}\right] \\ \begin{array}{cc} 1 & 2 \end{array} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array}.$$

Finally, applying the superposition method, we obtain the global stiffness matrix as follows (the matrix is **symmetric**, **banded** and **singular**)

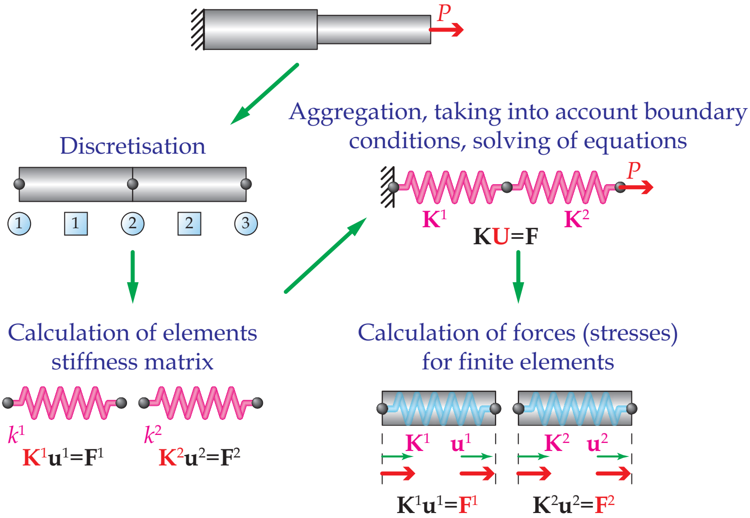
$$\mathbf{K} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} k_4 & -k_4 & 0 & 0 & 0 \\ -k_4 & k_1 + k_2 + k_4 & -k_2 & -k_1 & 0 \\ 0 & -k_2 & k_2 + k_3 & 0 & -k_3 \\ 0 & -k_1 & 0 & k_1 & 0 \\ 0 & 0 & -k_3 & 0 & k_3 \end{array}\right] \end{array}.$$



Applying boundary conditions and load ($U_4 = 0$, $U_5 = \Delta u$, $F_1 = T_1$, $F_2 = 0$, $F_3 = -T_2$) conditions, we obtain global FE equilibrium equations

$$\begin{bmatrix} k_4 & -k_4 & 0 & 0 & 0 \\ -k_4 & k_1 + k_2 + k_4 & -k_2 & -k_1 & 0 \\ 0 & -k_2 & k_2 + k_3 & 0 & -k_3 \\ 0 & -k_1 & 0 & k_1 & 0 \\ 0 & 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \equiv 0 \\ U_5 \equiv \Delta u \end{bmatrix} = \begin{bmatrix} F_1 \equiv T_1 \\ F_2 \equiv 0 \\ F_3 \equiv -T_2 \\ F_4 \\ F_5 \end{bmatrix}.$$

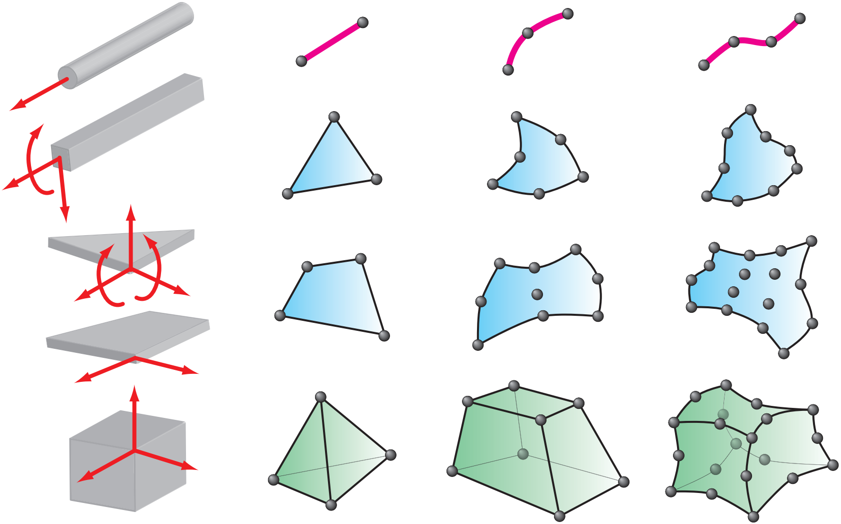
FEM procedure



FEM procedure in Structural Analysis:

- **discretisation** structure - divide structure ninto pieces,
- **determine FE equations** - describe the behaviour of the physical quantities on each element,
- **aggregation** - assemble the elements at the nodes to form an approximate system of equationa for the whole structure,
- **applying boundary conditions** - modification of the FE equations,
- **solving FE equations** - solving the system of equations involving unknown quantities (e.g., displacements),
- **calculate desired quantities** - at selected elements (e.g., strains and stresses).

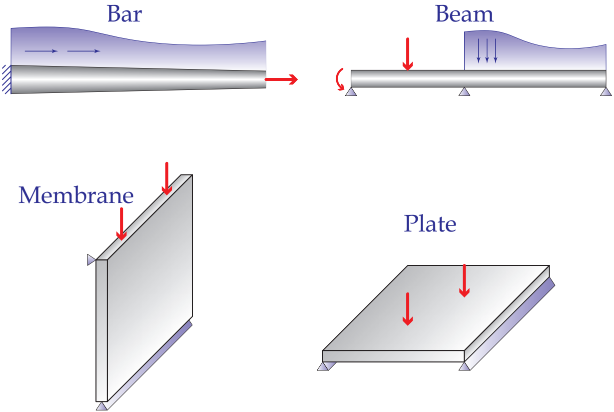
Finite element geometry



The geometry of the element is defined by the placement of the nodal points. Most elements used fairly simple geometries. In one-dimension, elements are usually straight lines or curved segments. In two dimensions they are of triangular or quadrilateral shape. In three dimensions the most common shapes are tetrahedron, hexahedron.

Finite element geometry is often associated with its shape functions.

Types of finite elements



Finite elements in structural mechanics are respect to the original physical structure. These elements are usually derived from Mechanics-of-Materials simplified theories or from the subdivision of structural components viewed as continua.

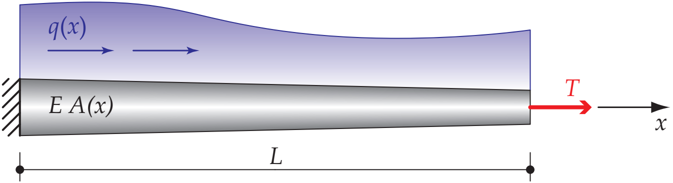
Elements has different sets of degrees of freedom (DOF). The degrees of freedom specify the state of the element. They also function as „handles" through which adjacent elements are connected. DOFs are defined as the values (and possibly derivatives) of a primary field variable at nodal points. For mechanical elements, the primary variable is the displacement field and the DOF for many (but not all) elements are the displacement components at the nodes.

Most structural analysis problems can be treated as **linear static** problems, based on the following assumptions:

- **small deformations** - loading pattern is not changed due to deformed shape,
- **elastic materials** - no plasticity or failures,
- **static loads** - the load is applied to the structure in a slow or steady fashion.

Linear analysis can provide most of the information about the behaviour of a structure and can be good approximation for many analyses. It is also bases of nonlinear analysis in most of the cases.

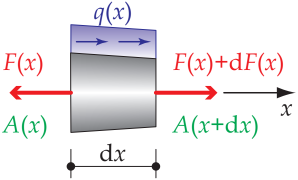
Consider a uniform prismatic bar with cross-sectional area $A(x)$ and length l. Material of bar has elastic-modulus E, $q(x)$ is distributed load and T is concentrated force.



This is **one-dimension problem**. One preferred dimension: **the longitudinal dimension** or **axial dimension** is much larger that the other two dimensions, which are collectively known as **transverse dimensions**. The intersection of a plane normal to the longitudinal dimension and the bar defines the cross sections. The longitudinal dimension defines the longitudinal axis x.

The bar resists an internal **axial force** (normal force) along its longitudinal dimension.

Using equilibrium requirements of infinitesimal part of bar dx, we have



$$-F(x) + F(x) + \mathrm{d}F(x) + q(x)\mathrm{d}x = 0,$$

where $F(x)$ and $F(x) + \mathrm{d}F(x)$ are **normal forces**, $A(x)$ and $A(x + \mathrm{d}x)$ are **cross section areas**.

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} + q(x) = 0,$$
$$0 < x < L.$$

Using stress-force relation

$$\sigma(x) = \frac{F(x)}{A(x)}, \quad F(x) = \sigma(x)A(x),$$

stress-strain relation (Hooke's law)

$$\sigma(x) = E\varepsilon(x),$$

and strain-displacement relation (Cauchy strain tensor)

$$\varepsilon(x) = \frac{\mathrm{d}u(x)}{\mathrm{d}x},$$

we state that

$$F(x) = EA(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}.$$

Using formula

$$F(x) = EA(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x},$$

our differential equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EA(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) + q(x) = 0,$$
$$0 < x < L,$$

if the rigidity is constant $EA = \text{const.}$, we have

$$EA\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + q(x) = 0,$$
$$0 < x < L.$$

Boundary conditions are

$$u(x = 0) = 0,$$
$$F(x = L) = EA\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_{x=L} = T \Rightarrow EA\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_{x=L} = T.$$

A boundary condition which specifies the value of the function itself is a **essential boundary condition**, or Dirichlet boundary condition (u has a known value).

A boundary condition which specifies the value of the normal derivative of the function is a **natural boundary condition**, or Neumann boundary condition ($\frac{\mathrm{d}u}{\mathrm{d}x}$ has a known value).

Finally, the boundary problem has a form

$$
\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}x}\left(EA(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)+q(x)=0,\\
0<x<L,\\
u(x=0)=0,\\
F(x=L)=EA\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_{x=L}=T.
\end{gathered}
$$

The finite element method starts by rewriting the local formulation in an equivalent variational form

$$\langle Au, v \rangle - \langle q, v \rangle .$$

$$-EA\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = q(x), \quad 0 < x < L, \quad u(x=0) = 0, \quad EA\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_{x=L} = T,$$

$$\Downarrow$$

$$\int_0^L -EA\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} v(x)\,\mathrm{d}x - \int_0^L q(x)v(x)\,\mathrm{d}x = 0.$$

Function $v(x)$ minimizes the functional and satisfies, at the same time, the differential equation (Galerkin formulation).

Application of integration by parts ($\int a'b = ab - \int ab', \quad a = -EA\frac{du}{dx}, \quad b' = \frac{dv}{dx}$) yields

$$\left[-EA\frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\right]_0^L - \int_0^L -EA\frac{\mathrm{d}u(x)}{\mathrm{d}x}\frac{\mathrm{d}v(x)}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L q(x)v(x)\,\mathrm{d}x,$$

$$\int_0^L EA\frac{\mathrm{d}u(x)}{\mathrm{d}x}\frac{\mathrm{d}v(x)}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L q(x)v(x)\,\mathrm{d}x + Tv(L).$$

The finite element method includes the boundary condition as integrals in a functional that is being minimized, so the construction procedure is independent of a particular boundary conditions of the problem.

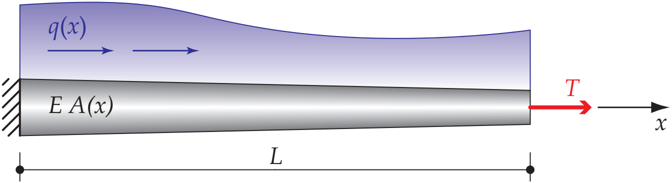
Above equation is called **Galerkin variational** or **weak form** of the problem defined by differential equation.

The finite element equations of the bar will be derived from the **minimum potential energy principle**.

In Mechanics of Materials it is shown that the **internal energy density** at a point of a linear-elastic material is

$$U = \int_V \frac{1}{2} \boldsymbol{\sigma}^{\mathsf{T}} \boldsymbol{\varepsilon} \, \mathrm{d}V.$$

For one-dimension state



$$U = \frac{1}{2} \int_V \sigma \varepsilon \, \mathrm{d}v.$$

This U is also called the **strain energy density**.

Stress σ is to be regarded as linked to the displacement u through Hooke's law and the strain-displacements relation

$$\sigma(x) = E\varepsilon(x),$$
$$\varepsilon(x) = \frac{\mathrm{d}u(x)}{\mathrm{d}x}.$$

Integration over the volume of the bar gives the total internal energy

$$U = \frac{1}{2}\int_V \sigma\varepsilon \, \mathrm{d}v = \frac{1}{2}\int_L EA(x)\varepsilon^2(x)\, \mathrm{d}x = \frac{1}{2}\int_L EA(x)\left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x.$$

If the rigidity is constant $EA = \text{const.}$, we have

$$U = \frac{1}{2}EA\int_L \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x,$$

in which all integrand quantities may depend on x.

The external energy (equal to work) due to applied mechanical loads pools contributions from two sources:

- the distributed load $q(x)$, this contributes a cross-section density of $q(x)u(x)$ because $q(x)$ is assumed to be already integrated over the section

$$L_q = \int_L q(x)u(x)\,\mathrm{d}x,$$

- any applied loads, the end load T would contribute $Tu(L)$

$$L_T = Tu(L).$$

Finally, in our case

$$W = L = L_q + L_T = \int_L q(x)u(x)\,\mathrm{d}x + Tu(L).$$

The **total potential energy** of the bar is given by

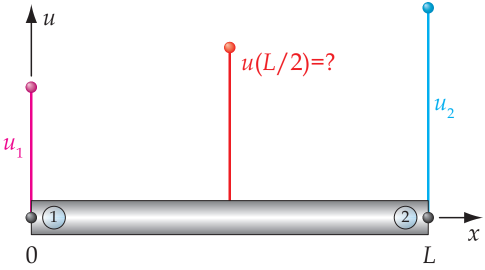
$$\Pi = U - W.$$

Mathematically this is a functional, it depends only on the axial displacement u(x). For bar with distributed load an one force

$$\Pi = U - L_q - L_T = \frac{1}{2}EA\int_L \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x - \int_L q(x)u(x)\mathrm{d}x - Tu(L).$$

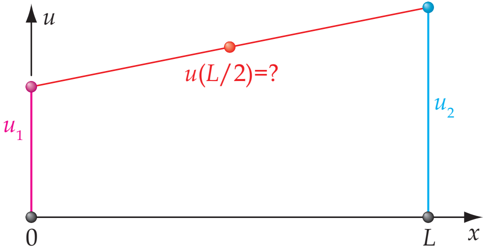
The solution to the problem formulated in this way will be the displacement function $u(x)$, for which the potential energy Π **reaches the minimum**.

After solving the FEM equations, we get the values of the displacements (or other unknowns) in nodes.



How to determine the displacement in the middle of a finite element?

- we can increase the number of finite elements so that the node is in the point of interest,
- we can set up some way of changing the displacement in the element and describe it with a function (**interpolation**).



In the second case, the simplest function can be the **linear** function in form $u(x) = ax + b$. For an element with a length of L, it must fulfil the following conditions:

- at the beginning of the finite element, for $x = 0$ must have the value u_1

$$u(0) = a \cdot 0 + b = u_1,$$

- at the end of the finite element, for $x = L$ must be u_2

$$u(L) = a \cdot L + b = u_2.$$

The above equations will allow us to determine the unknown coefficients a and b, which are equal

$$a = \frac{u_2 - u_1}{L},$$
$$b = u_1,$$

the displacement function in the element will take the form

$$u(x) = \frac{u_2 - u_1}{L} x + u_1.$$

Notes:

- it's easy to calculate such functions,
- each element will have a different approximation function,
- coefficient a has no physical interpretation.

Can functions be calculated differently (**better**)? Let's transform our function

$$u(x) = \frac{u_2 - u_1}{L}x + u_1 = \frac{u_2}{L}x - \frac{u_1}{L}x + u_1 = u_1\left(1 - \frac{1}{L}x\right) + u_2\left(\frac{1}{L}x\right),$$

let

$$N_1(x) = 1 - \frac{x}{L}, \quad N_2(x) = \frac{x}{L},$$

we can write

$$u(x) = N_1(x)u_1 + N_2(x)u_2.$$

Function $u(x)$ is a linear combination of expressions $N_1(x)$, $N_2(x)$ and nodal displacements u_1 i u_2.

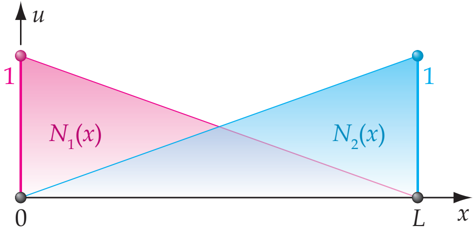
Let's check the properties of expressions $N_1(x)$ i $N_2(x)$:

- for $x = 0$

$$N_1(0) = 1 - \frac{0}{L} = 1, \quad N_2(0) = \frac{0}{L} = 0,$$

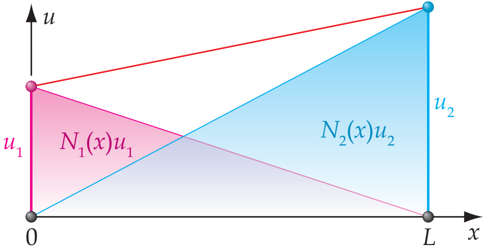
- for $x = L$

$$N_1(L) = 1 - \frac{L}{L} = 1 - 1 = 0, \quad N_2(L) = \frac{L}{L} = 1.$$



Functions $N_1(x)$ i $N_2(x)$ are linear Lagrange interpolation functions. In FEM interpolation functions are named **shape functions**.

Other types of interpolation functions are also used, for example Hermit polynomials, Serendip polynomials.



The general form of Lagrange interpolation can be written as

$$u_n(x) = \sum_{i=0}^{n} N_{n,i}(x)u_i = N_{n,0}(x)u_0 + N_{n,1}(x)u_1 + N_{n,2}(x)u_2 + \ldots + N_{n,n}(x)u_n,$$

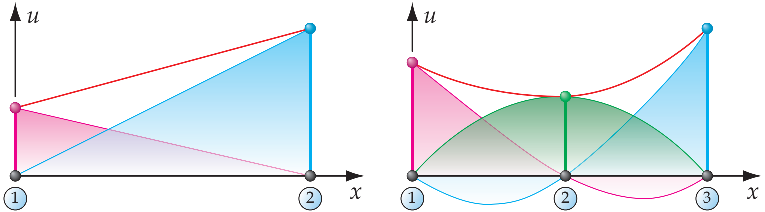
where n determines the degree of interpolative polynomial, and i is function number.

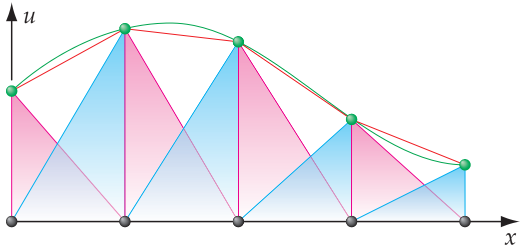
Notes:

- function $u(x)$ has transparent structure, it consists of similar parts,
- each part is the product of displacement in the node u_i (a value that has a physical interpretation) and the shape function $N_i(x)$,
- the value of the shape function $N_i(x)$ represents the displacement u_i contribution to the displacement value $u(x)$,
- functions $N_i(x)$ will be the same for all elements of the same type (depend only on its length),
- the functions $N_i(x)$ will be the same for each degree of freedom.

The Lagrange interpolation functions (Lagrange base functions) has a general form

$$N_{n,i}(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{(x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_{x+1})\dots(x - x_n)}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)},$$





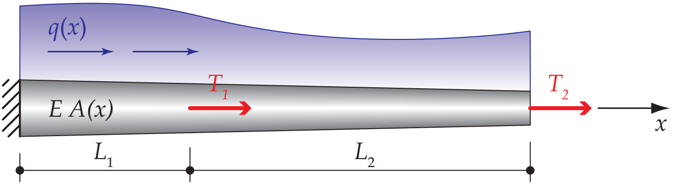
Notes:

- „broken lines" of interpolation in elements replace the real continuous displacement function,
- each continuous function can be approximated with any accuracy,
- the more elements, the better results.

Shape function $N_i(x)$ associated with node i of element must satisfy the following conditions:

- **interpolation condition** - takes a unit value at node i, and is zero at all other nodes,
- **local support condition** - vanishes over any element boundary (a side in 2D, a face in 3D) that does not include node i,
- **interelement compatibility condition** - satisfies C^0 continuity between adjacent elements over any element boundary that includes node i,
- **completeness condition** - the interpolation is able to represent exactly any displacement field which is a linear polynomial, in particular, a constant value.

Consider a uniform prismatic bar with cross-sectional area $A(x)$ and length L. $q(x)$ is distributed load, T_1 and T_2 are concentrated forces.



The total potential energy of the bar is given by

$$\Pi = U - L,$$

where U is strain energy

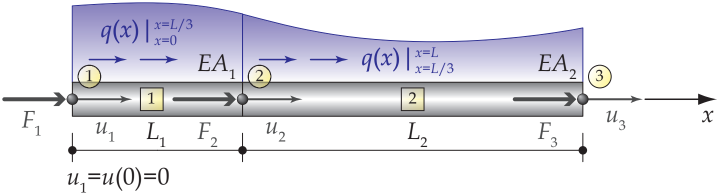
$$U = \frac{1}{2} E \int_0^L A(x) \left(u'(x)\right)^2 \mathrm{d}x,$$

and L is external energy (work)

$$L = \int_0^L q(x)u(x)\,\mathrm{d}x + \sum_{i=1}^{n} F_i Q_i.$$

F_i is generalized nodal force, Q_i is generalized nodal displacements (acting in the direction of these forces).

Taking into account the geometry and the boundary conditions, we can discretize bar in the following way



Lets define the global vector of degrees of freedom **Q** ($\{\bullet\}$ – denoting column vector, $\{\bullet\}^{\mathsf{T}} \equiv [\bullet]$)

$$\mathbf{Q} = \begin{Bmatrix} Q_1 & Q_2 & Q_3 \end{Bmatrix} \equiv \begin{Bmatrix} u_1 & u_2 & u_3 \end{Bmatrix},$$

and global vector of nodal forces **F**

$$\mathbf{F} = \begin{Bmatrix} F_1 & F_2 & F_3 \end{Bmatrix}.$$

Vector **Q** included:

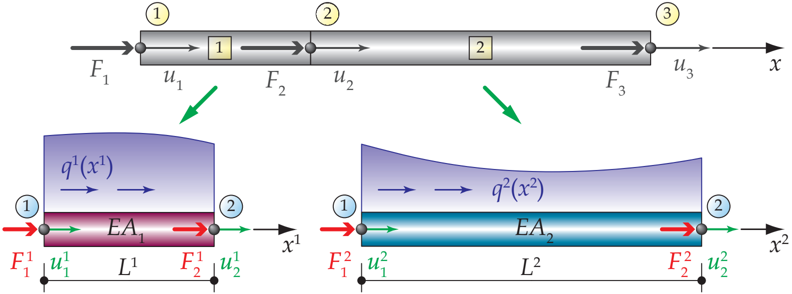
- essential boundary conditions $Q_1 \equiv u_1 = 0$,
- primary unknowns Q_2 i Q_3.

Vector **F** included:

- natural boundary conditions $F_2 = T_1$ i $F_3 = T_2$,
- secondary unknown F_1.

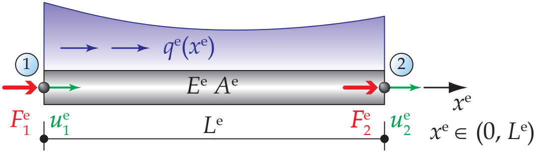
Total potential energy of the bar is the sum of the potential energy of the bar parts (finite elements)

$$\Pi = \sum_{e=1}^{m} \Pi^e = \Pi^1 + \Pi^2,$$



and we consider any finite element. For convenience, we define local coordinate systems related to elements x^e.

For finite element



we define vector of degrees of freedom $\mathbf{Q}^e$

$$\mathbf{Q}^e = \{Q^e_1 \quad Q^e_2\} \equiv \{u^e_1 \quad u^e_2\},$$

and vector of nodal forces $\mathbf{F}^e$

$$\mathbf{F}^e = \{F^e_1 \quad F^e_2\}.$$

We can write the displacements $u^e(x^e)$ as

$$u^e(x^e) = N^e_1(x^e)u^e_1 + N^e_2(x^e)u^e_2 = N^e_1(x^e)Q^e_1 + N^e_2(x^e)Q^e_2.$$

In matrix form

$$u^{\mathrm{e}}(x^{\mathrm{e}}) = \mathbf{N}^{\mathrm{e}}(x^{\mathrm{e}})\mathbf{Q}^{\mathrm{e}} = \left(\mathbf{Q}^{\mathrm{e}}\right)^{\mathrm{T}} \left(\mathbf{N}^{\mathrm{e}}(x^{\mathrm{e}})\right)^{\mathrm{T}},$$

where $\mathbf{N}^{\mathrm{e}}(x^{\mathrm{e}})$ can be written as

$$\mathbf{N}^{\mathrm{e}}(x^{\mathrm{e}}) = \begin{bmatrix} N_1^{\mathrm{e}}(x^{\mathrm{e}}) & N_2^{\mathrm{e}}(x^{\mathrm{e}}) \end{bmatrix},$$

and $N_1^{\mathrm{e}}(x^{\mathrm{e}})$ i $N_2^{\mathrm{e}}(x^{\mathrm{e}})$ are linear shape functions (Lagrange interpolation)

$$N_1^{\mathrm{e}}(x^{\mathrm{e}}) = 1 - \frac{x^{\mathrm{e}}}{L^{\mathrm{e}}}, \quad N_2^{\mathrm{e}}(x^{\mathrm{e}}) = \frac{x^{\mathrm{e}}}{L^{\mathrm{e}}}.$$

The derivative of the displacement function has the form

$$u'^{\mathrm{e}}(x^{\mathrm{e}}) = \mathbf{N}'^{\mathrm{e}}(x^{\mathrm{e}})\mathbf{Q}^{\mathrm{e}} = \left(\mathbf{Q}^{\mathrm{e}}\right)^{\mathrm{T}} \left(\mathbf{N}'^{\mathrm{e}}(x^{\mathrm{e}})\right)^{\mathrm{T}},$$

where

$$\mathbf{N}'^{\mathrm{e}}(x^{\mathrm{e}}) = \begin{bmatrix} N_1'^{\mathrm{e}}(x^{\mathrm{e}}) & N_2'^{\mathrm{e}}(x^{\mathrm{e}}) \end{bmatrix},$$

and

$$N_1'^{\mathrm{e}}(x^{\mathrm{e}}) = -\frac{1}{L^{\mathrm{e}}}, \quad N_2'^{\mathrm{e}}(x^{\mathrm{e}}) = \frac{1}{L^{\mathrm{e}}}.$$

For a constant cross-section of a finite element A^{e} (it can be different for each element), total potential energy of the finite element e is

$$\Pi^{\mathrm{e}} = \frac{1}{2} E^{\mathrm{e}} A^{\mathrm{e}} \int_0^{L^{\mathrm{e}}} \left(u'^{\mathrm{e}}(x^{\mathrm{e}})\right)^2 \mathrm{d}x^{\mathrm{e}} - \int_0^{L^{\mathrm{e}}} q^{\mathrm{e}}(x^{\mathrm{e}}) u^{\mathrm{e}}(x^{\mathrm{e}}) \,\mathrm{d}x^{\mathrm{e}} - F_1^{\mathrm{e}} Q_1^{\mathrm{e}} - F_2^{\mathrm{e}} Q_2^{\mathrm{e}}.$$

Using Lagrange interpolation, potential energy can be written as

$$\begin{aligned}
\Pi^{\mathrm{e}} &= \frac{1}{2} E^{\mathrm{e}} A^{\mathrm{e}} \int_0^{L^{\mathrm{e}}} (\mathbf{Q}^{\mathrm{e}})^{\mathrm{T}} (\mathbf{N}'^{\mathrm{e}})^{\mathrm{T}} \mathbf{N}'^{\mathrm{e}} \mathbf{Q}^{\mathrm{e}} \,\mathrm{d}x^{\mathrm{e}} - \int_0^{L^{\mathrm{e}}} (\mathbf{Q}^{\mathrm{e}})^{\mathrm{T}} (\mathbf{N}^{\mathrm{e}})^{\mathrm{T}} q^{\mathrm{e}} \,\mathrm{d}x^{\mathrm{e}} - (\mathbf{Q}^{\mathrm{e}})^{\mathrm{T}} \mathbf{F}^{\mathrm{e}} = \\
&= \frac{1}{2} (\mathbf{Q}^{\mathrm{e}})^{\mathrm{T}} \left\{ E^{\mathrm{e}} A^{\mathrm{e}} \int_0^{L^{\mathrm{e}}} (\mathbf{N}'^{\mathrm{e}})^{\mathrm{T}} \mathbf{N}'^{\mathrm{e}} \,\mathrm{d}x^{\mathrm{e}} \right\} \mathbf{Q}^{\mathrm{e}} - (\mathbf{Q}^{\mathrm{e}})^{\mathrm{T}} \left\{ \int_0^{L^{\mathrm{e}}} (\mathbf{N}^{\mathrm{e}})^{\mathrm{T}} q^{\mathrm{e}} \,\mathrm{d}x^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}} \right\} = \\
&= \frac{1}{2} (\mathbf{Q}^{\mathrm{e}})^{\mathrm{T}} \mathbf{K}^{\mathrm{e}} \mathbf{Q}^{\mathrm{e}} - (\mathbf{Q}^{\mathrm{e}})^{\mathrm{T}} \left\{ \mathbf{P}^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}} \right\}.
\end{aligned}$$

In formula

$$\Pi^{\mathrm{e}} = \frac{1}{2}(\mathbf{Q}^{\mathrm{e}})^{\mathrm{T}}\mathbf{K}^{\mathrm{e}}\mathbf{Q}^{\mathrm{e}} - (\mathbf{Q}^{\mathrm{e}})^{\mathrm{T}}\left\{\mathbf{P}^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}}\right\},$$

we denote:

- **stiffness matrix** of the bar finite element

$$\mathbf{K}^{\mathrm{e}} = E^{\mathrm{e}}A^{\mathrm{e}}\int_0^{L^{\mathrm{e}}}(\mathbf{N}'^{\mathrm{e}}(x^{\mathrm{e}}))^{\mathrm{T}}\mathbf{N}'^{\mathrm{e}}(x^{\mathrm{e}})\,\mathrm{d}x^{\mathrm{e}},$$

- **vector of equivalents of distributed load** of the bar finite element

$$\mathbf{P}^{\mathrm{e}} = \int_0^{L^{\mathrm{e}}}(\mathbf{N}^{\mathrm{e}})^{\mathrm{T}}q^{\mathrm{e}}(x^{\mathrm{e}})\,\mathrm{d}x^{\mathrm{e}},$$

- **nodal forces vector** of the bar finite element $\mathbf{F}^{\mathrm{e}}$,
- **degrees of freedom vector** (primary unknowns, nodal displacements) of the bar finite element $\mathbf{Q}^{\mathrm{e}}$.

Use of the interpolation $u^e(x^e)$ caused, that the total potential energy of an element is now a function of many (in our case two) variables $\Pi^e(Q_1^e, Q_2^e)$ and the equilibrium requirements (minimum of the potential energy) is as follows

$$\left.\begin{array}{l}\dfrac{\partial \Pi^e}{\partial Q_1^e} = 0\\ \dfrac{\partial \Pi^e}{\partial Q_2^e} = 0\end{array}\right\} \equiv \frac{\partial \Pi^e}{\partial \mathbf{Q}^e} = 0 \Rightarrow \frac{\partial}{\partial \mathbf{Q}^e}\left(\frac{1}{2}(\mathbf{Q}^e)^T \mathbf{K}^e \mathbf{Q}^e - (\mathbf{Q}^e)^T \left\{\mathbf{P}^e + \mathbf{F}^e\right\}\right) = 0,$$

which leads to the FEM equilibrium equations for the bar finite element

$$\mathbf{K}^e \mathbf{Q}^e = \mathbf{P}^e + \mathbf{F}^e,$$

or

$$\mathbf{K}^e \mathbf{Q}^e = \mathbf{R}^e,$$

where we define **total load vector** of finite element

$$\mathbf{R}^e = \mathbf{P}^e + \mathbf{F}^e.$$

If $E^e = \text{const.}$ and $A^e = \text{const.}$ are constant, we have

$$\mathbf{K}^e = E^e A^e \int_0^{L^e} (\mathbf{N}'^e(x^e))^T \mathbf{N}'^e(x^e)\, \mathrm{d}x^e = E^e A^e \int_0^{L^e} \begin{bmatrix} -\frac{1}{L^e} \\ \frac{1}{L^e} \end{bmatrix} \begin{bmatrix} -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix} \mathrm{d}x^e =$$

$$= E^e A^e \int_0^{L^e} \begin{bmatrix} \frac{1}{(L^e)^2} & -\frac{1}{(L^e)^2} \\ -\frac{1}{(L^e)^2} & \frac{1}{(L^e)^2} \end{bmatrix} \mathrm{d}x^e = E^e A^e \begin{bmatrix} \frac{x^e}{(L^e)^2} & -\frac{x^e}{(L^e)^2} \\ -\frac{x^e}{(L^e)^2} & \frac{x^e}{(L^e)^2} \end{bmatrix} \Bigg|_0^{L^e} =$$

$$= E^e A^e \begin{bmatrix} \frac{1}{L^e} & -\frac{1}{L^e} \\ -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix},$$

$$\mathbf{K}^e = \begin{bmatrix} \frac{E^e A^e}{L^e} & -\frac{E^e A^e}{L^e} \\ -\frac{E^e A^e}{L^e} & \frac{E^e A^e}{L^e} \end{bmatrix}. \tag{1}$$

For $q^{\mathrm{e}} = \text{const.}$

$$\mathbf{P}^{\mathrm{e}} = q^{\mathrm{e}} \int_0^{L^{\mathrm{e}}} (\mathbf{N}^{\mathrm{e}})^{\mathrm{T}} q^{\mathrm{e}} \,\mathrm{d}x^{\mathrm{e}} = q^{\mathrm{e}} \int_0^{L^{\mathrm{e}}} \begin{bmatrix} 1 - \dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}} \\ \dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}} \end{bmatrix} \mathrm{d}x^{\mathrm{e}} = q^{\mathrm{e}} \left. \begin{bmatrix} x^{\mathrm{e}} - \dfrac{(x^{\mathrm{e}})^2}{2L^{\mathrm{e}}} \\ \dfrac{(x^{\mathrm{e}})^2}{2L^{\mathrm{e}}} \end{bmatrix} \right|_0^{L^{\mathrm{e}}} = q^{\mathrm{e}} \begin{bmatrix} \dfrac{L^{\mathrm{e}}}{2} \\ \dfrac{L^{\mathrm{e}}}{2} \end{bmatrix},$$

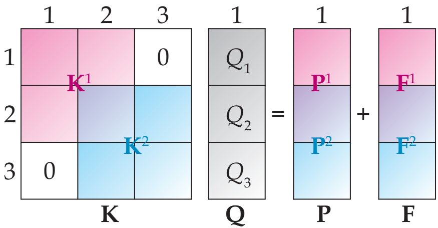
$$\mathbf{P}^{\mathrm{e}} = \begin{bmatrix} \dfrac{q^{\mathrm{e}} L^{\mathrm{e}}}{2} \\ \dfrac{q^{\mathrm{e}} L^{\mathrm{e}}}{2} \end{bmatrix}.$$

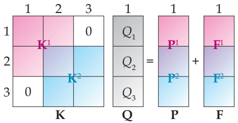
Elements connectivity table has a form

Element	Node 1	Node 2
1	1	2
2	2	3

The continuity conditions of displacements in common nodes of finite elements, allow us to write

$$\mathbf{Q} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} Q_1^1 \\ Q_2^1 = Q_1^2 \\ Q_2^2 \end{bmatrix}.$$





Global (structural) stiffness matrix is

$$\mathbf{K} = \mathop{\mathsf{A}}_{e=1}^{n} \mathbf{K}^{e} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{13} \\ K_{31} & K_{33} & K_{33} \end{bmatrix} = \begin{bmatrix} K_{11}^{1} & K_{12}^{1} & 0 \\ K_{21}^{1} & K_{22}^{1} + K_{11}^{2} & K_{12}^{2} \\ 0 & K_{21}^{2} & K_{22}^{2} \end{bmatrix}.$$

Global (structural) nodal forces vectors

$$\mathbf{P} = \mathop{\mathsf{A}}_{e=1}^{n} \mathbf{P}^{e} = \begin{bmatrix} P_{1} \\ P_{2} \\ P_{3} \end{bmatrix} = \begin{bmatrix} P_{1}^{1} \\ P_{2}^{1} + P_{1}^{2} \\ P_{2}^{2} \end{bmatrix},$$

$$\mathbf{F} = \mathop{\mathsf{A}}_{e=1}^{n} \mathbf{F}^{e} = \begin{bmatrix} F_{1} \\ F_{2} \\ F_{3} \end{bmatrix} = \begin{bmatrix} F_{1}^{1} \\ F_{2}^{1} + F_{1}^{2} \\ F_{2}^{2} \end{bmatrix}.$$

Global FEM equilibrium equations is

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 \\ 0 & K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} P_1^1 \\ P_2^1 + P_1^2 \\ P_2^2 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}.$$

Solution of the system of FEM equations requires consideration of boundary conditions

$$Q_1 = u(0) = 0, \quad F_2 = T_1, \quad F_3 = T_2,$$

problem equations become

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 \\ 0 & K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{bmatrix} 0 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} P_1^1 \\ P_2^1 + P_1^2 \\ P_2^2 \end{bmatrix} + \begin{bmatrix} F_1 \\ T_1 \\ T_2 \end{bmatrix}.$$

Solving FEM equations, we obtain nodal displacements Q_2 and Q_3 and nodal force F_1.

Using

$$\mathbf{Q} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} Q_1^1 \\ Q_2^1 = Q_1^2 \\ Q_2^2 \end{bmatrix},$$

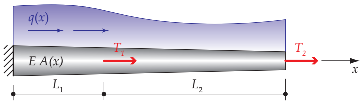
and interpolation

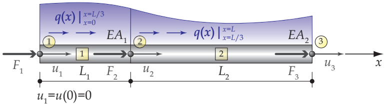
$$u^e(x^e) = N_1^e(x^e)Q_1^e + N_2^e(x^e)Q_2^e = \left(1 - \frac{x^e}{L^e}\right) Q_1^e + \left(\frac{x^e}{L^e}\right) Q_2^e,$$

we can write displacement functions

$$u(x) = \begin{cases} u^1(x^1) = \left(1 - \frac{x^1}{L^1}\right) Q_1 + \left(\frac{x^1}{L^1}\right) Q_2 & \text{for e} = 1, \\ u^2(x^2) = \left(1 - \frac{x^2}{L^2}\right) Q_2 + \left(\frac{x^2}{L^2}\right) Q_3 & \text{for e} = 2. \end{cases}$$

Analysis of a bar discretized with two finite elements





Example data:

- $L_1 = 0.4$ m,
- $L_2 = 0.8$ m,
- $A_1 = 0.5 \cdot 10^{-3}$ m^2,
- $A_2 = 0.4 \cdot 10^{-3}$ m^2,
- $E = 200$ GPa,
- $q = 10$ kN/m,
- $T_1 = 40$ kN,
- $T_2 = 5$ kN.

Element e $= 1$: $E^1 = 200$ GPa, $A^1 = 0.5 \cdot 10^{-3}$ m^2, $L^1 = 0.4$ m, $q^1 = 10$ kN/m,

$$\mathbf{K}^1 = 10^6 \begin{bmatrix} 250 & -250 \\ -250 & 250 \end{bmatrix}, \quad \mathbf{P}^1 = 10^3 \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Element e $= 2$: $E^2 = 200\,\text{GPa}$, $A^1 = 0.4 \cdot 10^{-3}\,\text{m}^2$, $L^2 = 0.8\,\text{m}$, $q^2 = 10\,{}^{\text{kN}}/_{\text{m}}$,

$$\mathbf{K}^2 = 10^6 \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}, \quad \mathbf{P}^2 = 10^3 \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

Global stiffness matrix and vectors

$$\mathbf{K} = 10^6 \begin{bmatrix} 250 & -250 & 0 \\ -250 & 350 & -100 \\ 0 & -100 & 100 \end{bmatrix}, \quad \mathbf{P} = 10^3 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix},$$

$$\mathbf{F} = 10^3 \begin{bmatrix} F_1 \\ 40 \\ 5 \end{bmatrix}.$$

A set of FEM equations (boundary conditions taking into account)

$$10^6 \begin{bmatrix} 250 & -250 & 0 \\ -250 & 350 & -100 \\ 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} 0 \\ Q_2 \\ Q_3 \end{bmatrix} = 10^3 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + 10^3 \begin{bmatrix} F_1 \\ 40 \\ 5 \end{bmatrix}.$$

From equations 2 and 3 we can calculate displacements

$$Q_2 = 0.00022\,\text{m}, \quad Q_3 = 0.00031\,\text{m}.$$

From equation 1 we can calculate force (reaction)

$$F_1 = -250 \cdot 10^6 \cdot Q_2 - 2 \cdot 10^3 = -250 \cdot 10^6 \cdot 0.00022 - 2 \cdot 10^3 = -57\,\text{kN}.$$

Global vector of nodal displacements is

$$\mathbf{Q} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.00022 \\ 0.00031 \end{bmatrix},$$

and vectors of nodal displacements for elements are

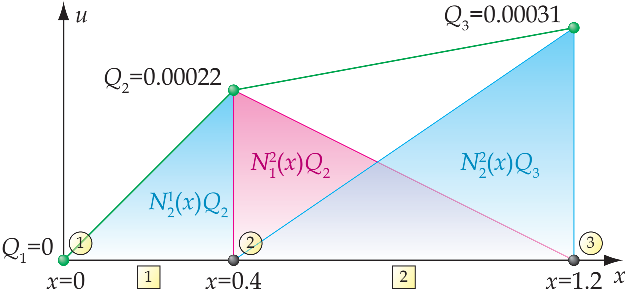
$$\mathbf{Q}^1 = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_1^1 \\ Q_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.00022 \end{bmatrix},$$

$$\mathbf{Q}^2 = \begin{bmatrix} Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} Q_1^2 \\ Q_2^2 \end{bmatrix} = \begin{bmatrix} 0.00022 \\ 0.00031 \end{bmatrix}.$$

Displacement function

$$u(x) = \begin{cases} u^1(x^1) = \left(\dfrac{x^1}{0.4}\right) 0.00022 & \text{for e} = 1, \\ u^2(x^2) = \left(1 - \dfrac{x^2}{0.8}\right) 0.00022 + \left(\dfrac{x^2}{0.8}\right) 0.00031 & \text{for e} = 2. \end{cases}$$

Displacement diagram



Using

$$F = A\sigma, \quad \sigma = E\varepsilon, \quad \varepsilon = \frac{\mathrm{d}u(x)}{\mathrm{d}x},$$

and

$$F = AE\frac{\mathrm{d}u(x)}{\mathrm{d}x},$$

forces function for both elements

$$F^1(x^1) = EA^1 u'^1(x) = EA^1(N_1'^1(x^1)Q_1 + N_2'^1(x^1)Q_2),$$
$$F^2(x^2) = EA^2 u'^2(x) = EA^2(N_1'^2(x^2)Q_2 + N_2'^2(x^2)Q_3).$$

Element e $= 1$, $L^1 = 0.4\,\mathrm{m}$, $Q_1 = 0\,\mathrm{m}$, $Q_2 = 0.00022\,\mathrm{m}$,

$$F^1(x^1) = EA^1\left(\left(-\frac{1}{L^1}\right)Q_1 + \left(\frac{1}{L^1}\right)Q_2\right) =$$

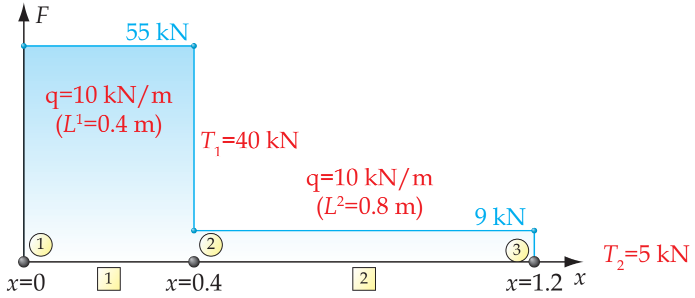
$$= 200\cdot 10^9\cdot 5\cdot 10^{-4}\left(-\left(\frac{1}{0.4}\right)0 + \left(\frac{1}{0.4}\right)0.00022\right) = 55\,\mathrm{kN}.$$

Element e $= 2$, $L^2 = 0.8\,\mathrm{m}$, $Q_2 = 0.00022\,\mathrm{m}$, $Q_3 = 0.00031\,\mathrm{m}$,

$$F^2(x^2) = EA^2\left(\left(-\frac{1}{L^2}\right)Q_2 + \left(\frac{1}{L^2}\right)Q_3\right) =$$

$$= 200\cdot 10^9\cdot 4\cdot 10^{-4}\left(-\left(\frac{1}{0.8}\right)0.00022 + \left(\frac{1}{0.8}\right)0.00031\right) = 9\,\mathrm{kN}.$$

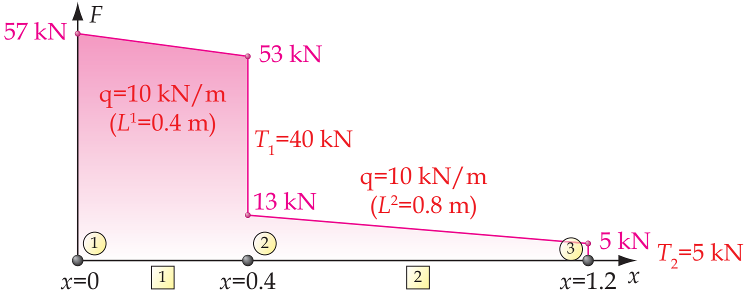
Forces diagram



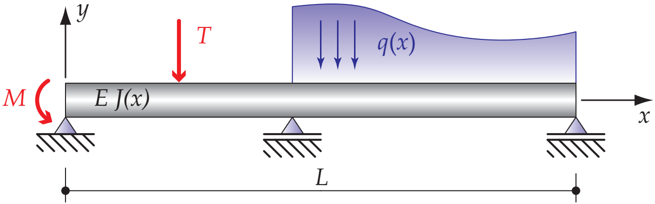
Caution! In the diagram forces sense are associated with the adopted coordinate system for finite elements, this has nothing to do with the marking convention often used in the mechanics of materials.

The results obtained by differentiating the displacement function are quite low quality, (e.g. they do not ensure continuity at the boundaries of finite elements). The quality of the solution can be improved by increasing the number of finite elements or by using the higher order base functions.

"Real forces" diagram

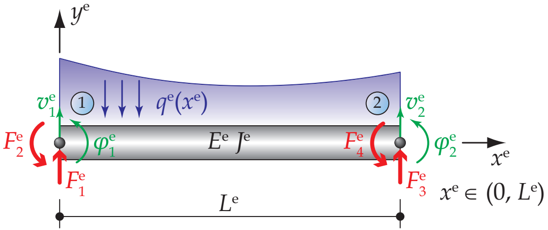


Self study example - beam finite element



The total potential energy of the beam is given by

$$\Pi = \frac{1}{2}E\int_0^L J(x)\left(v''(x)\right)^2 \,\mathrm{d}x - \int_0^L q(x)v(x)\,\mathrm{d}x + \sum_{i=1}^{n} F_i Q_i.$$



Vector of beam degree of freedom $\mathbf{Q}^e$ is

$$\mathbf{Q}^e = \left\{ Q_1^e \quad Q_2^e \quad Q_3^e \quad Q_4^e \right\} \equiv \left\{ v_1^e \quad \varphi_1^e \quad v_2^e \quad \varphi_2^e \right\},$$

and nodal force vectors $\mathbf{F}^e$ is given by

$$\mathbf{F}^e = \left\{ F_1^e \quad F_2^e \quad F_3^e \quad F_4^e \right\}.$$

Assuming a constant moment of inertia for the whole element, the total potential energy of the beam finite element is

$$\Pi^{e} = \frac{1}{2} E^{e} J^{e} \int_{0}^{L^{e}} \left(v^{\prime\prime e}(x^{e}) \right)^{2} \mathrm{d}x^{e} - \int_{0}^{L^{e}} q^{e}(x^{e}) v^{e}(x) \, \mathrm{d}x^{e} - F_{1}^{e} Q_{1}^{e} - F_{2}^{e} Q_{2}^{e} - F_{3}^{e} Q_{3}^{e} - F_{4}^{e} Q_{4}^{e}.$$

We use **Hermite interpolation** for displacement function

$$\begin{aligned} u^{e}(x^{e}) &= N_{1}^{e}(x^{e}) v_{1}^{e} + N_{2}^{e}(x^{e}) \varphi_{1}^{e} + N_{3}^{e}(x^{e}) v_{2}^{e} + N_{4}^{e}(x^{e}) \varphi_{2}^{e} = \\ &= N_{1}^{e}(x^{e}) Q_{1}^{e} + N_{2}^{e}(x^{e}) Q_{2}^{e} + N_{3}^{e}(x^{e}) Q_{3}^{e} + N_{4}^{e}(x^{e}) Q_{4}^{e} = \\ &= \mathbf{N}^{e}(x^{e}) \mathbf{Q}^{e} = \left(\mathbf{Q}^{e} \right)^{T} \left(\mathbf{N}^{e}(x^{e}) \right)^{T}, \end{aligned}$$

where $\mathbf{N}^{e}(x^{e})$ can be written as

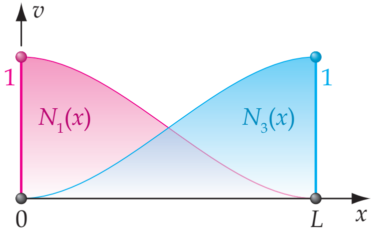
$$\mathbf{N}^{e}(x^{e}) = \begin{bmatrix} N_{1}^{e}(x^{e}) & N_{2}^{e}(x^{e}) & N_{3}^{e}(x^{e}) & N_{4}^{e}(x^{e}) \end{bmatrix},$$

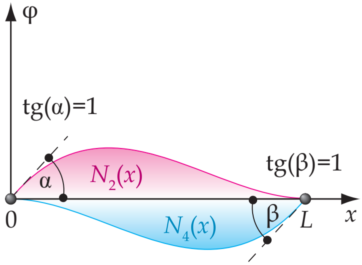
and $N_{1}^{e}(x^{e})$, $N_{2}^{e}(x^{e})$, $N_{3}^{e}(x^{e})$ i $N_{4}^{e}(x^{e})$ are shape functions.

Hermite shape functions have a form

$$N_1^e(x^e) = 1 - 3\left(\frac{x^e}{L^e}\right)^2 + 2\left(\frac{x^e}{L^e}\right)^3, \quad N_2^e(x^e) = x^e\left(1 - 2\left(\frac{x^e}{L^e}\right) + \left(\frac{x^e}{L^e}\right)^2\right),$$

$$N_3^e(x^e) = 3\left(\frac{x^e}{L^e}\right)^2 - 2\left(\frac{x^e}{L^e}\right)^3, \qquad N_4^e(x^e) = x^e\left(-\left(\frac{x^e}{L^e}\right) + \left(\frac{x^e}{L^e}\right)^2\right).$$





The second derivative of the displacement function take the form

$$v''^{\mathrm{e}}(x^{\mathrm{e}}) = \mathbf{N}''^{\mathrm{e}}(x^{\mathrm{e}})\mathbf{Q}^{\mathrm{e}} = (\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}} \left(\mathbf{N}''^{\mathrm{e}}(x^{\mathrm{e}})\right)^{\mathsf{T}},$$

where

$$\mathbf{N}''^{\mathrm{e}}(x^{\mathrm{e}}) = \begin{bmatrix} N_1''^{\mathrm{e}}(x^{\mathrm{e}}) & N_2''^{\mathrm{e}}(x^{\mathrm{e}}) & N_3''^{\mathrm{e}}(x^{\mathrm{e}}) & N_4''^{\mathrm{e}}(x^{\mathrm{e}}) \end{bmatrix},$$

and

$$N_1''^{\mathrm{e}}(x^{\mathrm{e}}) = -\frac{6}{(L^{\mathrm{e}})^2} + \frac{12}{(L^{\mathrm{e}})^3}x^{\mathrm{e}}, \qquad N_2''^{\mathrm{e}}(x^{\mathrm{e}}) = -\frac{4}{L^{\mathrm{e}}} + \frac{6}{(L^{\mathrm{e}})^2}x^{\mathrm{e}},$$

$$N_3''^{\mathrm{e}}(x^{\mathrm{e}}) = \frac{6}{(L^{\mathrm{e}})^2} - \frac{12}{(L^{\mathrm{e}})^3}x^{\mathrm{e}}, \qquad N_4''^{\mathrm{e}}(x^{\mathrm{e}}) = -\frac{2}{L^{\mathrm{e}}} + \frac{6}{(L^{\mathrm{e}})^2}x^{\mathrm{e}}.$$

Using interpolation, potential energy can be written as

$$\begin{aligned}
\Pi^{\mathrm{e}} &= \frac{1}{2}E^{\mathrm{e}}J^{\mathrm{e}}\int_0^{L^{\mathrm{e}}}(\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}}(\mathbf{N}''^{\mathrm{e}})^{\mathsf{T}}\mathbf{N}''^{\mathrm{e}}\mathbf{Q}^{\mathrm{e}}\,\mathrm{d}x^{\mathrm{e}} - \int_0^{L^{\mathrm{e}}}(\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}}(\mathbf{N}^{\mathrm{e}})^{\mathsf{T}}q^{\mathrm{e}}\,\mathrm{d}x^{\mathrm{e}} - (\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}}\mathbf{F}^{\mathrm{e}} = \\
&= \frac{1}{2}(\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}}\left\{E^{\mathrm{e}}J^{\mathrm{e}}\int_0^{L^{\mathrm{e}}}(\mathbf{N}''^{\mathrm{e}})^{\mathsf{T}}\mathbf{N}''^{\mathrm{e}}\,\mathrm{d}x^{\mathrm{e}}\right\}\mathbf{Q}^{\mathrm{e}} - (\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}}\left\{\int_0^{L^{\mathrm{e}}}(\mathbf{N}^{\mathrm{e}})^{\mathsf{T}}q^{\mathrm{e}}\,\mathrm{d}x^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}}\right\} = \\
&= \frac{1}{2}(\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}}\mathbf{K}^{\mathrm{e}}\mathbf{Q}^{\mathrm{e}} - (\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}}\left\{\mathbf{P}^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}}\right\}.
\end{aligned}$$

In formula

$$\Pi^{\mathrm{e}} = \frac{1}{2}(\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}}\mathbf{K}^{\mathrm{e}}\mathbf{Q}^{\mathrm{e}} - (\mathbf{Q}^{\mathrm{e}})^{\mathsf{T}}\left\{\mathbf{P}^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}}\right\},$$

we denote:

- **stiffness matrix** of the beam finite element

$$\mathbf{K}^{\mathrm{e}} = E^{\mathrm{e}}A^{\mathrm{e}}\int_{0}^{L^{\mathrm{e}}}(\mathbf{N}''^{\mathrm{e}}(x^{\mathrm{e}}))^{\mathsf{T}}\mathbf{N}''^{\mathrm{e}}(x^{\mathrm{e}})\,\mathrm{d}x^{\mathrm{e}},$$

- **vector of equivalents of distributed load** of the beam finite element

$$\mathbf{P}^{\mathrm{e}} = \int_{0}^{L^{\mathrm{e}}}(\mathbf{N}^{\mathrm{e}})^{\mathsf{T}}q^{\mathrm{e}}(x^{\mathrm{e}})\,\mathrm{d}x^{\mathrm{e}},$$

- **nodal forces vector** of the bar finite element $\mathbf{F}^{\mathrm{e}}$,
- **degrees of freedom vector** (primary unknowns, nodal displacements) of the bem finite element $\mathbf{Q}^{\mathrm{e}}$.

Use of the interpolation $u^e(x^e)$ caused, that the total potential energy of an element is now a function of four variables $\Pi^e(Q_1^e, Q_2^e, Q_3^e, Q_4^e)$ and the equilibrium requirements (minimum of the potential energy) is as follows

$$\left.\begin{aligned} \frac{\partial \Pi^e}{\partial Q_1^e} &= 0 \\ \frac{\partial \Pi^e}{\partial Q_2^e} &= 0 \\ \frac{\partial \Pi^e}{\partial Q_3^e} &= 0 \\ \frac{\partial \Pi^e}{\partial Q_4^e} &= 0 \end{aligned}\right\} \equiv \frac{\partial \Pi^e}{\partial \mathbf{Q}^e} = 0 \Rightarrow \frac{\partial}{\partial \mathbf{Q}^e}\left(\frac{1}{2}(\mathbf{Q}^e)^T \mathbf{K}^e \mathbf{Q}^e - (\mathbf{Q}^e)^T \left\{\mathbf{P}^e + \mathbf{F}^e\right\}\right) = 0,$$

which leads to the FEM equilibrium equations for the beam finite element

$$\mathbf{K}^e \mathbf{Q}^e = \mathbf{P}^e + \mathbf{F}^e,$$

or

$$\mathbf{K}^e \mathbf{Q}^e = \mathbf{R}^e, \quad \mathbf{R}^e = \mathbf{P}^e + \mathbf{F}^e,$$

where we define **total load vector** of finite element $\mathbf{R}^e$.

If $E^e = \text{const.}$ and $J^e = \text{const.}$ are constant, we have

$$\mathbf{K}^e = E^e J^e \int_0^{L^e} (\mathbf{N}''^e(x^e))^T \mathbf{N}''^e(x^e)\,\mathrm{d}x^e =$$

$$= E^e J^e \int_0^{L^e} \begin{bmatrix} -\frac{6}{(L^e)^2} + \frac{12x^e}{(L^e)^3} \\ -\frac{4}{L^e} + \frac{6x^e}{(L^e)^2} \\ \frac{6}{(L^e)^2} - \frac{12x^e}{(L^e)^3} \\ -\frac{2}{L^e} + \frac{6x^e}{(L^e)^2} \end{bmatrix} \left[-\frac{6}{(L^e)^2} + \frac{12x^e}{(L^e)^3} \quad -\frac{4}{L^e} + \frac{6x^e}{(L^e)^2} \right.$$

$$\left. \frac{6}{(L^e)^2} - \frac{12x^e}{(L^e)^3} \quad -\frac{2}{L^e} + \frac{6x^e}{(L^e)^2} \right] \mathrm{d}x^e,$$

after integration, **stiffens matrix** of beam finite element is

$$\mathbf{K}^{\mathrm{e}} = \begin{bmatrix} \frac{12E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^3} & \frac{6E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^2} & -\frac{12E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^3} & \frac{6E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^2} \\ \frac{6E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^2} & \frac{4E^{\mathrm{e}}J^{\mathrm{e}}}{L^{\mathrm{e}}} & -\frac{6E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^2} & \frac{2E^{\mathrm{e}}J^{\mathrm{e}}}{L^{\mathrm{e}}} \\ -\frac{12E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^3} & -\frac{6E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^2} & \frac{12E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^3} & -\frac{6E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^2} \\ \frac{6E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^2} & \frac{2E^{\mathrm{e}}J^{\mathrm{e}}}{L^{\mathrm{e}}} & -\frac{6E^{\mathrm{e}}J^{\mathrm{e}}}{(L^{\mathrm{e}})^2} & \frac{4E^{\mathrm{e}}J^{\mathrm{e}}}{L^{\mathrm{e}}} \end{bmatrix}.$$

For $q^{\mathrm{e}} = \text{const.}$

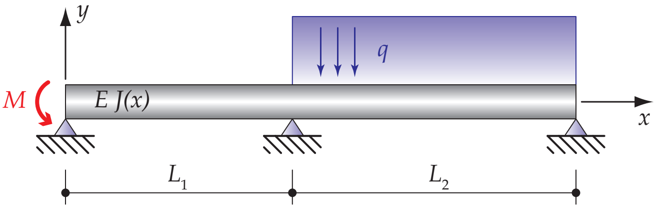
$$\mathbf{P}^{\mathrm{e}} = q^{\mathrm{e}} \int_0^{L^{\mathrm{e}}} (\mathbf{N}^{\mathrm{e}})^{\mathrm{T}} q^{\mathrm{e}} \,\mathrm{d}x^{\mathrm{e}} = q^{\mathrm{e}} \int_0^{L^{\mathrm{e}}} \begin{bmatrix} 1 - 3\left(\dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}}\right)^2 + 2\left(\dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}}\right)^3 \\ x^{\mathrm{e}}\left(1 - 2\left(\dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}}\right) + \left(\dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}}\right)^2\right) \\ 3\left(\dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}}\right)^2 - 2\left(\dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}}\right)^3 \\ x^{\mathrm{e}}\left(-\left(\dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}}\right) + \left(\dfrac{x^{\mathrm{e}}}{L^{\mathrm{e}}}\right)^2\right) \end{bmatrix} \mathrm{d}x^{\mathrm{e}},$$

after integration

$$\mathbf{P}^{\mathrm{e}} = \begin{bmatrix} \dfrac{q^{\mathrm{e}} L^{\mathrm{e}}}{2} \\ \dfrac{q^{\mathrm{e}} (L^{\mathrm{e}})^2}{12} \\ \dfrac{q^{\mathrm{e}} L^{\mathrm{e}}}{2} \\ -\dfrac{q^{\mathrm{e}} (L^{\mathrm{e}})^2}{12} \end{bmatrix}.$$

Further proceedings: aggregation, taking into account of boundary conditions, solving of equations are analogous to procedure described for bar elements.

Analysis of a beam discretized with two finite elements

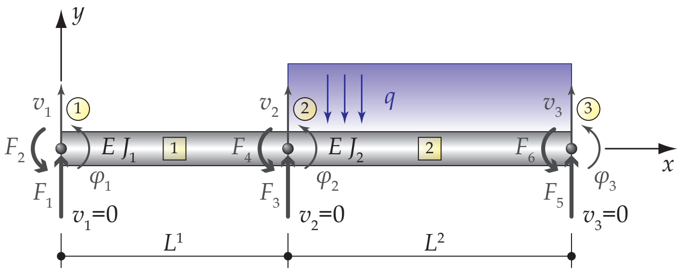


Example data:

- $J_1 = 5 \cdot 10^{-5}$ m^4 (I-beam 260 × 135 mm, web thickness: 7 mm, flange thickness: 10 mm),
- $J_2 = 1 \cdot 10^{-4}$ m^4 (I-beam 330 × 160 mm, web thickness: 7 mm, flange thickness: 10 mm),

- $L_1 = 6$ m,
- $L_2 = 8$ m,

- $E = 200$ GPa,
- $q = 10$ kN/m,
- $M = 20$ kNm.

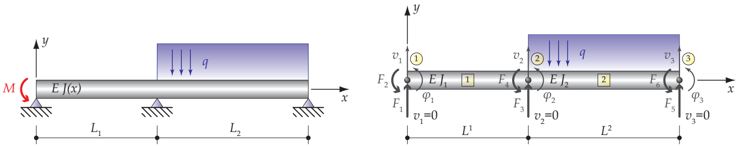


Lets define the global vector of degrees of freedom

$$\mathbf{Q} = \begin{Bmatrix} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{Bmatrix} \equiv \begin{Bmatrix} v_1 & \varphi_1 & v_2 & \varphi_2 & v_3 & \varphi_3 \end{Bmatrix},$$

and global vector of nodal forces

$$\mathbf{F} = \begin{Bmatrix} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 \end{Bmatrix}.$$

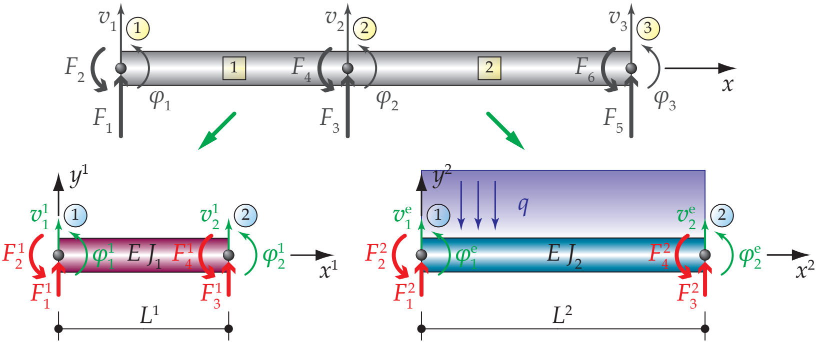


Vector **Q** included:

- essential boundary conditions $Q_1 \equiv v_1 = Q_3 \equiv v_2 = Q_5 \equiv v_3 = 0$,
- primary unknowns Q_2, Q_4 i Q_6.

Vector **F** included:

- natural boundary conditions $F_2 = M$, $F_4 = 0$ i $F_6 = 0$,
- secondary unknowns F_1, F_3 i F_5.



Elements connectivity table has form

Element	Node 1	Node 2
1	1	2
2	2	3

Global (structural) stiffness matrix is

$$\begin{bmatrix} K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} & 0 & 0 \\ K^1_{21} & K^1_{22} & K^1_{23} & K^1_{24} & 0 & 0 \\ K^1_{31} & K^1_{32} & K^1_{33}+K^2_{11} & K^1_{34}+K^2_{12} & K^2_{13} & K^2_{14} \\ K^1_{41} & K^1_{42} & K^1_{43}+K^2_{21} & K^1_{44}+K^2_{22} & K^2_{23} & K^2_{24} \\ 0 & 0 & K^2_{31} & K^2_{32} & K^2_{33} & K^2_{34} \\ 0 & 0 & K^2_{41} & K^2_{42} & K^2_{43} & K^2_{44} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} P^1_1 \\ P^1_2 \\ P^1_3+P^2_1 \\ P^1_4+P^2_2 \\ P^2_3 \\ P^2_4 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}.$$

Solution of the system of FEM equations requires consideration of boundary conditions

$$Q_1 = Q_3 = Q_5 = 0, \quad F_2 = M, \quad F_4 = F_6 = 0,$$

problem equations become

$$\begin{bmatrix} K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} & 0 & 0 \\ K^1_{21} & K^1_{22} & K^1_{23} & K^1_{24} & 0 & 0 \\ K^1_{31} & K^1_{32} & K^1_{33}+K^2_{11} & K^1_{34}+K^2_{12} & K^2_{13} & K^2_{14} \\ K^1_{41} & K^1_{42} & K^1_{43}+K^2_{21} & K^1_{44}+K^2_{22} & K^2_{23} & K^2_{24} \\ 0 & 0 & K^2_{31} & K^2_{32} & K^2_{33} & K^2_{34} \\ 0 & 0 & K^2_{41} & K^2_{42} & K^2_{43} & K^2_{44} \end{bmatrix} \begin{bmatrix} 0 \\ Q_2 \\ 0 \\ Q_4 \\ 0 \\ Q_6 \end{bmatrix} = \begin{bmatrix} P^1_1 \\ P^1_2 \\ P^1_3+P^2_1 \\ P^1_4+P^2_2 \\ P^2_3 \\ P^2_4 \end{bmatrix} + \begin{bmatrix} F_1 \\ M \\ F_3 \\ 0 \\ F_5 \\ 0 \end{bmatrix}.$$

Element e $= 1$: $E^1 = 200$ GPa, $J^1 = 5 \cdot 10^{-5}$ m^4, $L^1 = 6$ m, $q^1 = 0$,

$$\mathbf{K}^1 = 10^5 \begin{bmatrix} 5.556 & 16.167 & -5.556 & 16.667 \\ 16.167 & 66.667 & -16.167 & 33.333 \\ -5.556 & -16.167 & 5.556 & -16.167 \\ 16.167 & 33.333 & -16.167 & 66.667 \end{bmatrix}, \quad \mathbf{P}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Element e $= 2$: $E^2 = 200$ GPa, $J^2 = 1 \cdot 10^{-4}$ m^4, $L^2 = 8$ m, $q^2 = 10$ kN/m,

$$\mathbf{K}^2 = 10^5 \cdot \begin{bmatrix} 4.688 & 18.75 & -4.688 & 18.75 \\ 18.750 & 100.00 & -18.750 & 50.00 \\ -4.688 & -18.75 & 4.688 & -18.75 \\ 18.750 & 50.00 & -18.750 & 100.00 \end{bmatrix}, \quad \mathbf{P}^2 = 10^4 \begin{bmatrix} -4.000 \\ -5.333 \\ -4.000 \\ 5.333 \end{bmatrix}.$$

After aggregation we have

$$10^5 \cdot \begin{bmatrix} 5.556 & 16.167 & -5.556 & 16.667 & 0 & 0 \\ 16.167 & 66.667 & -16.167 & 33.333 & 0 & 0 \\ -5.556 & -16.667 & 10.243 & 2.083 & -4.688 & 18.750 \\ 16.667 & 33.333 & 2.083 & 166.667 & -18.750 & 50.000 \\ 0 & 0 & -4.688 & -18.750 & 4.688 & -18.750 \\ 0 & 0 & 18.750 & 50.000 & -18.750 & 100.000 \end{bmatrix} \begin{bmatrix} 0 \\ Q_2 \\ 0 \\ Q_4 \\ 0 \\ Q_6 \end{bmatrix} =$$

$$= 10^4 \cdot \begin{bmatrix} 0 \\ 0 \\ -4.000 \\ -5.333 \\ -4.000 \\ 5.333 \end{bmatrix} + \begin{bmatrix} F_1 \\ 20 \\ F_3 \\ 0 \\ F_5 \\ 0 \end{bmatrix}.$$

From equation 2, 4 i 6 we can calculate displacements

$$10^5 \cdot \begin{bmatrix} 66.667 & 33.333 & 0 \\ 33.333 & 166.667 & 50.000 \\ 0 & 50.000 & 100.000 \end{bmatrix} \begin{bmatrix} Q_2 \\ Q_4 \\ Q_6 \end{bmatrix} = 10^4 \cdot \begin{bmatrix} 2.000 \\ -5.333 \\ 5.333 \end{bmatrix},$$

$$Q_2 = 0.0066\,\text{rad}, \quad Q_4 = -0.0072\,\text{rad}, \quad Q_6 = 0.00893\,\text{rad}.$$

From equations 1, 3 i 5 we can calculate forces

$$F_1 = 16.667 \cdot 10^5 \cdot Q_2 + 16.667 \cdot 10^5 \cdot Q_4 = -1\,\text{kN},$$
$$F_3 = -16.667 \cdot 10^5 \cdot Q_2 + 2.083 \cdot 10^5 \cdot Q_4 + 18.75 \cdot 10^5 \cdot Q_6 + 4 \cdot 10^4 =$$
$$= 44.25\,\text{kNm},$$
$$F_5 = -18.75 \cdot 10^5 \cdot Q_4 - 18.75 \cdot 10^5 \cdot Q_6 + 4 \cdot 10^4 = 36.75\,\text{kN}.$$

Global vector of nodal displacements is

$$\mathbf{Q} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.0066 \\ 0 \\ -0.0072 \\ 0 \\ 0.00893 \end{bmatrix},$$

and vectors nodal displacements for elements are

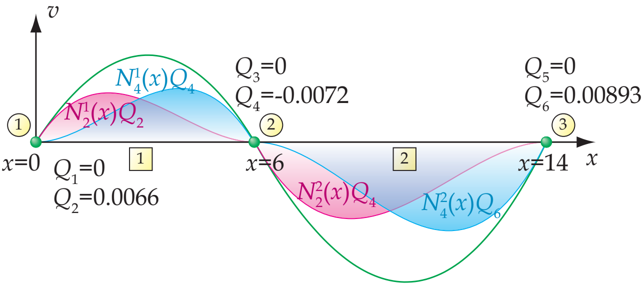
$$\mathbf{Q}^1 = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.0066 \\ 0 \\ -0.0072 \end{bmatrix},$$

$$\mathbf{Q}^2 = \begin{bmatrix} Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} Q_1^2 \\ Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.0072 \\ 0 \\ 0.00893 \end{bmatrix}.$$

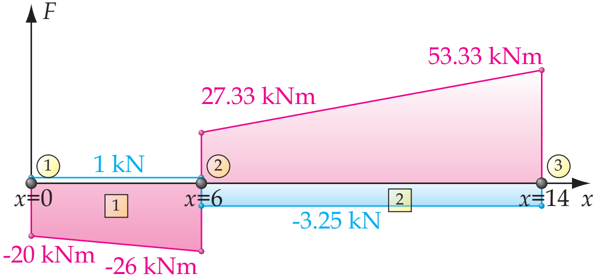
Displacement function

$$v(x) = \begin{cases} v^1(x^1) = x^1\left(1 - 2\left(\frac{x^1}{6}\right) + \left(\frac{x^1}{6}\right)^2\right)0.0066 + \\ \qquad + x^1\left(-\left(\frac{x^1}{6}\right) + \left(\frac{x^1}{6}\right)^2\right)(-0.0072) & \text{for e} = 1, \\ v^2(x^2) = x^2\left(1 - 2\left(\frac{x^2}{8}\right) + \left(\frac{x^2}{8}\right)^2\right)(-0.0072) + \\ \qquad + x^2\left(-\left(\frac{x^2}{8}\right) + \left(\frac{x^2}{8}\right)^2\right)0.00893 & \text{for e} = 2. \end{cases}$$

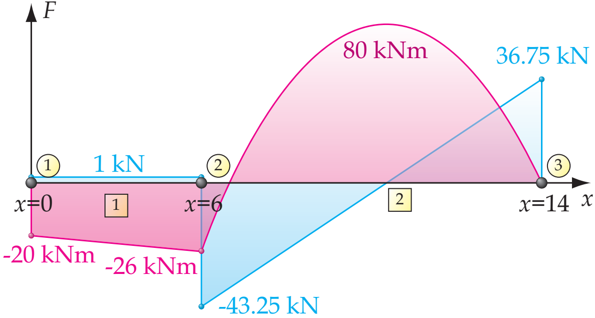
Displacement diagram



Forces diagram

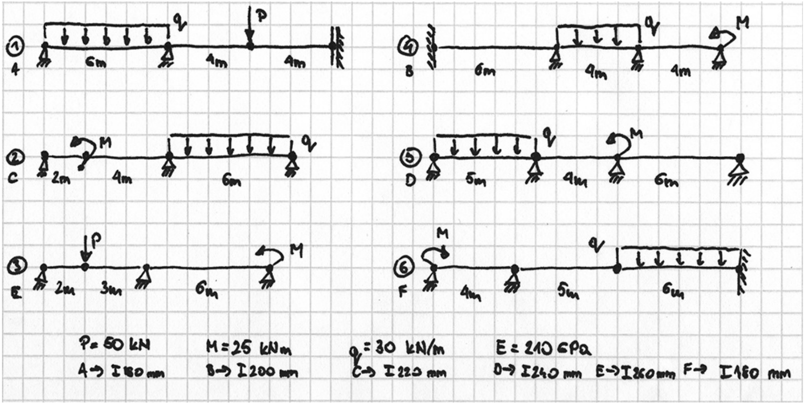


"Real forces" diagram



Final project

Extend the **SimpleFEM** program (available from the website `www.tu.kielce.pl/~sk/erasmus`) for a linear analysis of beams.



Students wishing to enhance knowledge of the Finite Elements Method should be interested in the following topics:

- Elements in Space.
- Two-Dimensional Problems.
- Isoparametric Representation.
- FEM Convergence Requirements.
- Solving FEM Equations.
- Geometrical and Physical Non-Linearity.
- Stability Analysis.
- Dynamic Analysis.